Quantization of systems with $OSp(2|2)$ symmetry

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Abstract

We study the quantization of systems that contain both ordinary fields with a positive norm and their counterparts obeying different statistics. The systems have novel fermionic symmetries different from the space-time supersymmetry and the BRST symmetry. The unitarity of systems holds by imposing subsidiary conditions on states.

1 Introduction

The spin-statistics theorem explains that *observed particles of integer spin obey Bose-Einstein statistics and are quantized by the commutation relations, and those of half odd integer spin obey Fermi-Dirac statistics and are quantized by the anti-commutation relations in the framework of relativistic quantum field theory* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The study on abnormal fields has been little carried out [14, 15, 16, 17], except for Faddeev-Popov ghosts, i.e., ghost fields appearing on the quantization of systems with local symmetries [18]. Here, abnormal fields mean particles obeying different statistics from ordinary ones. We refer to a scalar field following anti-commutation relations as a 'fermionic scalar field' and to a spinor field following commutation relations as a 'bosonic spinor field'.

The reasons for the indifference of abnormal fields would be as follows. First, they seem unrealistic because the standard model does not contain abnormal ones irrelevant to gauge symmetries. Second, in the introduction of abnormal fields, states with a negative norm appear and the unitarity of systems can be violated. Third, even if such unfavorable states are projected out by imposing subsidiary conditions on states, abnormal fields become unphysical and cannot give any effects on physical processes. Hence, we suppose that the existence of abnormal fields cannot be verified directly or this is the same as the non-existence.

Nevertheless, it would be meaningful to examine systems with abnormal fields from following reasons. There is a possibility that unphysical objects exist in nature if they

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are not prohibited from the consistency of theories. This is a similar idea to that Dirac predicted the existence of magnetic monopole based on quantum theory. Unphysical ones might play a vital role at a more fundamental level. Furthermore, it is expected that they might leave some fingerprints and we could check them as indirect proofs.

This paper takes a scholarly look at the nature of abnormal fields. We study the quantization of systems that contain both ordinary fields with a positive norm and their counterparts obeying different statistics. We find that the systems have fermionic symmetries and the unitarity of systems holds by imposing subsidiary conditions on states. The fermionic symmetries are novel ones on a space of quantum fields, different from the space-time supersymmetry and the BRST symmetry.

The content of this paper are as follows. We study the quantization of system of scalar fields with $OSp(2|2)$ symmetry in Sect. II and spinor fields with fermionic symmetries in Sect. III. Section IV is devoted to conclusions and discussions.

2 Systems of scalar fields with $OSp(2|2)$ symmetry

Let us study the system that an ordinary complex scalar field φ and the fermionic one c_φ coexist, described by the Lagrangian density,

$$\mathscr{L}_{\varphi,c_\varphi} = \partial_\mu\varphi^\dagger\partial^\mu\varphi - m^2\varphi^\dagger\varphi + \partial_\mu c_\varphi^\dagger\partial^\mu c_\varphi - m^2 c_\varphi^\dagger c_\varphi. \tag{1}$$

Based on the formulation with the property that *the hermitian conjugate of canonical momentum for a variable is just the canonical momentum for the hermitian conjugate of the variable*, we define the conjugate momentum of φ, $\varphi^\dagger$, c_φ and $c_\varphi^\dagger$ as

$$\pi \equiv \left(\frac{\partial\mathscr{L}_{\varphi,c_\varphi}}{\partial\dot{\varphi}}\right)_{\mathrm{R}} = \dot{\varphi}^\dagger,\ \ \pi^\dagger \equiv \left(\frac{\partial\mathscr{L}_{\varphi,c_\varphi}}{\partial\dot{\varphi}^\dagger}\right)_{\mathrm{L}} = \dot{\varphi}, \tag{2}$$

$$\pi_{c_\varphi} \equiv \left(\frac{\partial\mathscr{L}_{\varphi,c_\varphi}}{\partial\dot{c}_\varphi}\right)_{\mathrm{R}} = \dot{c}_\varphi^\dagger,\ \ \pi_{c_\varphi}^\dagger \equiv \left(\frac{\partial\mathscr{L}_{\varphi,c_\varphi}}{\partial\dot{c}_\varphi^\dagger}\right)_{\mathrm{L}} = \dot{c}_\varphi, \tag{3}$$

where R and L stand for the right-differentiation and the left-differentiation, respectively.

By solving the Klein-Gordon equations $\left(\Box + m^2\right)\varphi = 0$ and $\left(\Box + m^2\right)c_\varphi = 0$, we obtain the solutions

$$\varphi(x) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\left(a(\boldsymbol{k})e^{-ikx} + b^\dagger(\boldsymbol{k})e^{ikx}\right), \tag{4}$$

$$\varphi^\dagger(x) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\left(a^\dagger(\boldsymbol{k})e^{ikx} + b(\boldsymbol{k})e^{-ikx}\right), \tag{5}$$

$$\pi(x) = i\int d^3k\sqrt{\frac{k_0}{2(2\pi)^3}}\left(a^\dagger(\boldsymbol{k})e^{ikx} - b(\boldsymbol{k})e^{-ikx}\right), \tag{6}$$

$$\pi^\dagger(x) = -i\int d^3k\sqrt{\frac{k_0}{2(2\pi)^3}}\left(a(\boldsymbol{k})e^{-ikx} - b^\dagger(\boldsymbol{k})e^{ikx}\right), \tag{7}$$

$$c_\varphi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}} \left(c(\boldsymbol{k}) e^{-ikx} + d^\dagger(\boldsymbol{k}) e^{ikx} \right), \tag{8}$$

$$c_\varphi^\dagger(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}} \left(c^\dagger(\boldsymbol{k}) e^{ikx} + d(\boldsymbol{k}) e^{-ikx} \right), \tag{9}$$

$$\pi_{c_\varphi}(x) = i \int d^3k \sqrt{\frac{k_0}{2(2\pi)^3}} \left(c^\dagger(\boldsymbol{k}) e^{ikx} - d(\boldsymbol{k}) e^{-ikx} \right), \tag{10}$$

$$\pi_{c_\varphi}^\dagger(x) = -i \int d^3k \sqrt{\frac{k_0}{2(2\pi)^3}} \left(c(\boldsymbol{k}) e^{-ikx} - d^\dagger(\boldsymbol{k}) e^{ikx} \right), \tag{11}$$

where $k_0 = \sqrt{\boldsymbol{k}^2 + m^2}$ and $kx = k^\mu x_\mu$.

Using (2) and (3), the Hamiltonian density is obtained as

$$\begin{aligned} \mathscr{H}_{\varphi,c_\varphi} &= \pi\dot{\varphi} + \dot{\varphi}^\dagger \pi^\dagger + \pi_{c_\varphi} \dot{c}_\varphi + \dot{c}_\varphi^\dagger \pi_{c_\varphi}^\dagger - \mathscr{L}_{\varphi,c_\varphi} \\ &= \pi\pi^\dagger + \nabla\varphi^\dagger \nabla\varphi + m^2 \varphi^\dagger \varphi + \pi_{c_\varphi} \pi_{c_\varphi}^\dagger + \nabla c_\varphi^\dagger \nabla c_\varphi + m^2 c_\varphi^\dagger c_\varphi. \end{aligned} \tag{12}$$

The system is quantized by regarding variables as operators and imposing the following relations on the canonical pairs (φ, π), $(\varphi^\dagger, \pi^\dagger)$, $(c_\varphi, \pi_{c_\varphi})$ and $(c_\varphi^\dagger, \pi_{c_\varphi}^\dagger)$,

$$[\varphi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}), \ \ [\varphi^\dagger(\boldsymbol{x}, t), \pi^\dagger(\boldsymbol{y}, t)] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}), \tag{13}$$

$$\{c_\varphi(\boldsymbol{x}, t), \pi_{c_\varphi}(\boldsymbol{y}, t)\} = i\delta^3(\boldsymbol{x} - \boldsymbol{y}), \ \ \{c_\varphi^\dagger(\boldsymbol{x}, t), \pi_{c_\varphi}^\dagger(\boldsymbol{y}, t)\} = -i\delta^3(\boldsymbol{x} - \boldsymbol{y}), \tag{14}$$

where $[O_1, O_2] \equiv O_1 O_2 - O_2 O_1$, $\{O_1, O_2\} \equiv O_1 O_2 + O_2 O_1$, and only the non-vanishing ones are denoted. Or equivalently, the following relations are imposed on,

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{l})] = \delta^3(\boldsymbol{k} - \boldsymbol{l}), \ \ [b(\boldsymbol{k}), b^\dagger(\boldsymbol{l})] = \delta^3(\boldsymbol{k} - \boldsymbol{l}), \tag{15}$$

$$\{c(\boldsymbol{k}), c^\dagger(\boldsymbol{l})\} = \delta^3(\boldsymbol{k} - \boldsymbol{l}), \ \ \{d(\boldsymbol{k}), d^\dagger(\boldsymbol{l})\} = -\delta^3(\boldsymbol{k} - \boldsymbol{l}), \tag{16}$$

and others are zero.

By inserting (4) – (11) into (12), the Hamiltonian H_{φ,c_φ} is written by

$$H_{\varphi,c_\varphi} = \int \mathscr{H}_{\varphi,c_\varphi} d^3x = \int d^3k\, k_0 \left(a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + b^\dagger(\boldsymbol{k}) b(\boldsymbol{k}) + c^\dagger(\boldsymbol{k}) c(\boldsymbol{k}) - d^\dagger(\boldsymbol{k}) d(\boldsymbol{k}) \right). \tag{17}$$

Note that the sum of the zero-point energies vanishes due to the cancellation between contributions from $(\varphi, \varphi^\dagger)$ and $(c_\varphi, c_\varphi^\dagger)$.

The eigenstates for H_{φ,c_φ} are constructed by acting the creation operators $a^\dagger(\boldsymbol{k})$, $b^\dagger(\boldsymbol{k})$, $c^\dagger(\boldsymbol{k})$ and $d^\dagger(\boldsymbol{k})$ on the vacuum state $|0\rangle$, where $|0\rangle$ is defined by the conditions $a(\boldsymbol{k})|0\rangle = 0$, $b(\boldsymbol{k})|0\rangle = 0$, $c(\boldsymbol{k})|0\rangle = 0$ and $d(\boldsymbol{k})|0\rangle = 0$. We find that the energy is positive semi-definite, because the effect on the negative sign appearing in front of $d^\dagger(\boldsymbol{k}) d(\boldsymbol{k})$ in H_{φ,c_φ} changes into an opposite one by the negative sign in the relation $\{d(\boldsymbol{k}), d^\dagger(\boldsymbol{l})\} = -\delta^3(\boldsymbol{k} - \boldsymbol{l})$.

The microscopic causality also holds seen from the 4-dimensional relations as

$$[\varphi(x), \varphi^\dagger(y)] = \{c_\varphi(x), c_\varphi^\dagger(y)\} = \int \frac{d^3k}{(2\pi)^3 2k_0} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right)$$

$$= \int \frac{d^4 k}{(2\pi)^3} \epsilon(k_0)\delta(k^2 - m^2) e^{-ik(x-y)} \equiv i\varDelta(x-y), \tag{18}$$

$$[\varphi(x), \varphi(y)] = 0, \ \ [\varphi^\dagger(x), \varphi^\dagger(y)] = 0, \ \ \{c_\varphi(x), c_\varphi(y)\} = 0, \ \ \{c_\varphi^\dagger(x), c_\varphi^\dagger(y)\} = 0, \tag{19}$$

$$[\varphi(x), c_\varphi(y)] = 0, \ \ [\varphi(x), c_\varphi^\dagger(y)] = 0, \ \ [\varphi^\dagger(x), c_\varphi(y)] = 0, \ \ [\varphi^\dagger(x), c_\varphi^\dagger(y)] = 0, \tag{20}$$

where $\epsilon(k_0) = k_0/|k_0|$ with $\epsilon(0) = 0$, $\varDelta(x-y)$ is the invariant delta function, and two fields separated by a space-like interval commute or anti-commute with each other as seen from the relation $\varDelta(x-y) = 0$ for $(x-y)^2 < 0$. Note that bosonic variables composed of c_φ and $c_\varphi^\dagger$ are commutative to any bosonic variables separated by a space-like interval.

The system contains negative norm states originated from $\{d(\boldsymbol{k}), d^\dagger(\boldsymbol{l})\} = -\delta^3(\boldsymbol{k} - \boldsymbol{l})$. For instance, from the relation,

$$\begin{aligned} 0 < \int d^3 k \left| f(\boldsymbol{k}) \right|^2 &= -\int d^3 k \int d^3 l f(\boldsymbol{k})^* f(\boldsymbol{l}) \langle 0 | \{d(\boldsymbol{k}), d^\dagger(\boldsymbol{l})\} | 0 \rangle \\ &= -\int d^3 k \int d^3 l f(\boldsymbol{k})^* f(\boldsymbol{l}) \langle 0 | d(\boldsymbol{k}) d^\dagger(\boldsymbol{l}) | 0 \rangle = -\left| \int d^3 k f(\boldsymbol{k}) d^\dagger(\boldsymbol{k}) | 0 \rangle \right|^2, \end{aligned} \tag{21}$$

we see that the state $\int d^3 k f(\boldsymbol{k}) d^\dagger(\boldsymbol{k}) |0\rangle$ has a negative norm. Here, $f(\boldsymbol{k})$ is some square integrable functions. In the presence of negative norm states, the probability interpretation cannot be endured. In the following, it is shown that the system has fermionic symmetries and they can guarantee the unitarity of the system.

Now, let us investigate the symmetries of the system. The $\mathscr{L}_{\varphi, c_\varphi}$ is invariant under the transformations whose generators are the Lie algebras of $OSp(2|2)$. In the appendix A, we explain more about $OSp(2|2)$ and $OSp(1,1|2)$ and field theories with such symmetries.

The transformations form following types.

(a) $U(1)$ transformation relating φ and $\varphi^\dagger$:

$$\delta_{\rm o}\varphi = -iq\epsilon_{\rm o}\varphi, \ \ \delta_{\rm o}\varphi^\dagger = iq\epsilon_{\rm o}\varphi^\dagger, \ \ \delta_{\rm o} c_\varphi = 0, \ \ \delta_{\rm o} c_\varphi^\dagger = 0, \tag{22}$$

where q is a $U(1)$ charge of φ and $\epsilon_{\rm o}$ is an infinitesimal real number.

(b) $U(1)$ transformation relating c_φ and $c_\varphi^\dagger$:

$$\delta_{\rm g}\varphi = 0, \ \ \delta_{\rm g}\varphi^\dagger = 0, \ \ \delta_{\rm g} c_\varphi = -iq\epsilon_{\rm g} c_\varphi, \ \ \delta_{\rm g} c_\varphi^\dagger = iq\epsilon_{\rm g} c_\varphi^\dagger, \tag{23}$$

where q is a $U(1)$ charge of c_φ and $\epsilon_{\rm g}$ is an infinitesimal real number.

(c)Fermionic transformations:

$$\delta_{\rm F}\varphi = -r\zeta c_\varphi, \ \ \delta_{\rm F}\varphi^\dagger = 0, \ \ \delta_{\rm F} c_\varphi = 0, \ \ \delta_{\rm F} c_\varphi^\dagger = r\zeta\varphi^\dagger, \tag{24}$$

$$\delta_{\rm F}^\dagger\varphi = 0, \ \ \delta_{\rm F}^\dagger\varphi^\dagger = r\zeta^\dagger c_\varphi^\dagger, \ \ \delta_{\rm F}^\dagger c_\varphi = r\zeta^\dagger\varphi, \ \ \delta_{\rm F}^\dagger c_\varphi^\dagger = 0, \tag{25}$$

where $r = q^{1/2}$ and ζ and $\zeta^\dagger$ are Grassmann numbers. Note that $\delta_{\rm F}$ and $\delta_{\rm F}^\dagger$ are not generated by hermitian operators, different from the generator of the BRST transformation in systems with first class constraints [19] and that of the topological symmetry [20, 21].

From the above transformation properties, we see that $\boldsymbol{\delta}_{\mathrm{F}}$ and $\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}$ are nilpotent, i.e., ${\boldsymbol{\delta}_{\mathrm{F}}}^2 = 0$ and ${\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}}^2 = 0$ where $\boldsymbol{\delta}_{\mathrm{F}}$ and $\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}$, are defined by $\delta_{\mathrm{F}} = \zeta\boldsymbol{\delta}_{\mathrm{F}}$ and $\delta_{\mathrm{F}}^{\dagger} = \zeta^{\dagger}\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}$, respectively. Furthermore, the following algebraic relations hold:

$$
{Q_{\mathrm{F}}}^2 = 0, \;\; {Q_{\mathrm{F}}^{\dagger}}^2 = 0, \;\; \{Q_{\mathrm{F}}, Q_{\mathrm{F}}^{\dagger}\} = Q_{\mathrm{o}} + Q_{\mathrm{g}} \equiv N_{\mathrm{D}}, \tag{26}
$$

where Q_{F}, $Q_{\mathrm{F}}^{\dagger}$, Q_{o} and Q_{g} are corresponding generators (charges) given by

$$
\delta_{\mathrm{F}}\Phi = i[\zeta Q_{\mathrm{F}}, \Phi], \;\; \delta_{\mathrm{F}}^{\dagger}\Phi = i[Q_{\mathrm{F}}^{\dagger}\zeta^{\dagger}, \Phi], \;\; \delta_{\mathrm{o}}\Phi = i[\epsilon_{\mathrm{o}} Q_{\mathrm{o}}, \Phi], \;\; \delta_{\mathrm{g}}\Phi = i[\epsilon_{\mathrm{g}} Q_{\mathrm{g}}, \Phi]. \tag{27}
$$

From the definition,

$$
\zeta Q_{\mathrm{F}} \equiv \int d^3x \left[\left(\frac{\partial \mathscr{L}_{\varphi, c_{\varphi}}}{\partial \dot{\varphi}} \right)_{\mathrm{R}} \delta_{\mathrm{F}}\varphi + \delta_{\mathrm{F}} c_{\varphi}^{\dagger} \left(\frac{\partial \mathscr{L}_{\varphi, c_{\varphi}}}{\partial \dot{c}_{\varphi}^{\dagger}} \right)_{\mathrm{L}} \right], \tag{28}
$$

$$
Q_{\mathrm{F}}^{\dagger}\zeta^{\dagger} \equiv \int d^3x \left[\delta_{\mathrm{F}}^{\dagger}\varphi^{\dagger} \left(\frac{\partial \mathscr{L}_{\varphi, c_{\varphi}}}{\partial \dot{\varphi}^{\dagger}} \right)_{\mathrm{L}} + \left(\frac{\partial \mathscr{L}_{\varphi, c_{\varphi}}}{\partial \dot{c}_{\varphi}} \right)_{\mathrm{R}} \delta_{\mathrm{F}}^{\dagger} c_{\varphi} \right], \tag{29}
$$

the conserved fermionic charges Q_{F} and $Q_{\mathrm{F}}^{\dagger}$ are obtained by

$$
Q_{\mathrm{F}} = \int d^3x \, r \left(-\pi c_{\varphi} + \varphi^{\dagger}\pi_{c_{\varphi}}^{\dagger} \right) = -i \int d^3k \, r \left(a^{\dagger}(\boldsymbol{k}) c(\boldsymbol{k}) - d^{\dagger}(\boldsymbol{k}) b(\boldsymbol{k}) \right), \tag{30}
$$

$$
Q_{\mathrm{F}}^{\dagger} = \int d^3x \, r \left(-c_{\varphi}^{\dagger}\pi^{\dagger} + \pi_{c_{\varphi}}\varphi \right) = i \int d^3k \, r \left(c^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k}) - b^{\dagger}(\boldsymbol{k}) d(\boldsymbol{k}) \right). \tag{31}
$$

Then, under the fermionic transformations, the canonical momenta are transformed as,

$$
\delta_{\mathrm{F}}\pi = 0, \;\; \delta_{\mathrm{F}}\pi^{\dagger} = -r\zeta\pi_{c_{\varphi}}^{\dagger}, \;\; \delta_{\mathrm{F}}\pi_{c_{\varphi}} = r\zeta\pi, \;\; \delta_{\mathrm{F}}\pi_{c_{\varphi}}^{\dagger} = 0, \tag{32}
$$

$$
\delta_{\mathrm{F}}^{\dagger}\pi = r\zeta^{\dagger}\pi_{c_{\varphi}}, \;\; \delta_{\mathrm{F}}^{\dagger}\pi^{\dagger} = 0, \;\; \delta_{\mathrm{F}}^{\dagger}\pi_{c_{\varphi}} = 0, \;\; \delta_{\mathrm{F}}^{\dagger}\pi_{c_{\varphi}}^{\dagger} = -r\zeta^{\dagger}\pi^{\dagger}. \tag{33}
$$

The conserved $U(1)$ charge N_{D} is given by

$$
N_{\mathrm{D}} = \int d^3k \, q \left(a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k}) - b^{\dagger}(\boldsymbol{k}) b(\boldsymbol{k}) + c^{\dagger}(\boldsymbol{k}) c(\boldsymbol{k}) + d^{\dagger}(\boldsymbol{k}) d(\boldsymbol{k}) \right). \tag{34}
$$

We find that the $U(1)$ charge of particle corresponding $b^{\dagger}(\boldsymbol{k})|0\rangle$ and $d^{\dagger}(\boldsymbol{k})|0\rangle$ is opposite to that corresponding $a^{\dagger}(\boldsymbol{k})|0\rangle$ and $c^{\dagger}(\boldsymbol{k})|0\rangle$. Hence, $a(\boldsymbol{k})$ ($c(\boldsymbol{k})$) and $b^{\dagger}(\boldsymbol{k})$ ($d^{\dagger}(\boldsymbol{k})$) are regarded as the annihilation operator of particle (fermionic one) and the creation operator of antiparticle (antiparticle of fermionic one), respectively.

It is easily understood that $\mathscr{L}_{\varphi, c_{\varphi}}$ is invariant under the transformations (24) and (25), from the nilpotency of $\boldsymbol{\delta}_{\mathrm{F}}$ and $\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}$ and the relations,

$$
\mathscr{L}_{\varphi, c_{\varphi}} = \boldsymbol{\delta}_{\mathrm{F}}\boldsymbol{\delta}_{\mathrm{F}}^{\dagger} \left(\mathscr{L}_{\varphi}/q \right) = -\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}\boldsymbol{\delta}_{\mathrm{F}} \left(\mathscr{L}_{\varphi}/q \right), \tag{35}
$$

where $\mathscr{L}_{\varphi}$ is given by

$$
\mathscr{L}_{\varphi} = \partial_{\mu}\varphi^{\dagger}\partial^{\mu}\varphi - m^2\varphi^{\dagger}\varphi. \tag{36}
$$

The Hamiltonian density $\mathscr{H}_{\varphi,c_\varphi}$ is written in the $Q_{\rm F}$ and $Q_{\rm F}^\dagger$ exact forms such that

$$\mathscr{H}_{\varphi,c_\varphi} = \left\{Q_{\rm F}, \left\{Q_{\rm F}^\dagger, \mathscr{H}_\varphi / q\right\}\right\} = -\left\{Q_{\rm F}^\dagger, \left\{Q_{\rm F}, \mathscr{H}_\varphi / q\right\}\right\}, \tag{37}$$

where $\mathscr{H}_\varphi$ is given by

$$\mathscr{H}_\varphi = \pi\pi^\dagger + \nabla\varphi^\dagger\nabla\varphi + m^2\varphi^\dagger\varphi. \tag{38}$$

To formulate our model in a consistent manner, we use a feature that *a conserved charge can be, in general, set to be zero as a subsidiary condition.* We impose the following subsidiary conditions on states to select physical states,

$$Q_{\rm F}|{\rm phys}\rangle = 0,\ \ Q_{\rm F}^\dagger|{\rm phys}\rangle = 0,\ \ N_{\rm D}|{\rm phys}\rangle = 0. \tag{39}$$

Note that $Q_{\rm F}^\dagger|{\rm phys}\rangle = 0$ means $\langle{\rm phys}|Q_{\rm F} = 0$. The conditions (39) are interpreted as counterparts of the Kugo-Ojima subsidiary condition in the BRST quantization [22, 23]. We find that all states, except for the vacuum state $|0\rangle$, are unphysical because they do not satisfy (39). This feature is understood as the quartet mechanism [22, 23]. The projection operator $P^{(n)}$ on the states with n particles is given by

$$P^{(n)} = \frac{1}{n}\left(a^\dagger P^{(n-1)} a + b^\dagger P^{(n-1)} b + c^\dagger P^{(n-1)} c - d^\dagger P^{(n-1)} d\right) \quad (n \geq 1), \tag{40}$$

and is written by

$$P^{(n)} = i\left\{Q_{\rm F}, R^{(n)}\right\}\ , \tag{41}$$

where $R^{(n)}$ is given by

$$R^{(n)} = \frac{1}{n}\left(c^\dagger P^{(n-1)} a + b^\dagger P^{(n-1)} d\right) \quad (n \geq 1). \tag{42}$$

We find that any state with $n \geq 1$ is unphysical from the relation $\langle{\rm phys}|P^{(n)}|{\rm phys}\rangle = 0$ for $n \geq 1$. Then, we understand that both φ and c_φ become unphysical, and only $|0\rangle$ is the physical one. This is also regarded as a field theoretical version of the Parisi-Sourlas mechanism [24].

The system is also described by hermitian fermionic charges defined by $Q_1 \equiv Q_{\rm F} + Q_{\rm F}^\dagger$ and $Q_2 \equiv i(Q_{\rm F} - Q_{\rm F}^\dagger)$. They satisfy the relations $Q_1Q_2 + Q_2Q_1 = 0$, ${Q_1}^2 = N_{\rm D}$ and ${Q_2}^2 = N_{\rm D}$. Though Q_1, Q_2 and $N_{\rm D}$ form elements of the $N = 2$ (quantum mechanical) supersymmetry algebra [25], our system does not possess the space-time supersymmetry because $N_{\rm D}$ is not our Hamiltonian H_{φ,c_φ} but the $U(1)$ charge $N_{\rm D}$. Only the vacuum state is selected as the physical states by imposing the following subsidiary conditions on states, in place of (39),

$$Q_1|{\rm phys}\rangle = 0,\ \ Q_2|{\rm phys}\rangle = 0,\ \ N_{\rm D}|{\rm phys}\rangle = 0. \tag{43}$$

It is also understood that our fermionic symmetries are different from the space-time supersymmetry, from the fact that Q_1 and Q_2 are scalar charges. They are also different from the BRST symmetry, as seen from the algebraic relations among charges.

We discuss interactions among fields forming $Q_{\rm F}$-doublets. Let us consider a system with two sets of $Q_{\rm F}$-doublet scalar fields $(\varphi_1,\ c_{\varphi_1})$ and $(\varphi_2,\ c_{\varphi_2})$, described by the Lagrangian density,

$$\begin{aligned}\mathscr{L}_{\varphi_i,c_{\varphi_i}} &= \partial_\mu\varphi_1^\dagger\partial^\mu\varphi_1 - m_1^2\varphi_1^\dagger\varphi_1 + \partial_\mu c_{\varphi_1}^\dagger\partial^\mu c_{\varphi_1} - m_1^2 c_{\varphi_1}^\dagger c_{\varphi_1} \\ &\quad + \partial_\mu\varphi_2^\dagger\partial^\mu\varphi_2 - m_2^2\varphi_2^\dagger\varphi_2 + \partial_\mu c_{\varphi_2}^\dagger\partial^\mu c_{\varphi_2} - m_2^2 c_{\varphi_2}^\dagger c_{\varphi_2} \\ &\quad - \lambda\left(\varphi_1^\dagger\varphi_1 + c_{\varphi_1}^\dagger c_{\varphi_1}\right)\left(\varphi_2^\dagger\varphi_2 + c_{\varphi_2}^\dagger c_{\varphi_2}\right) \\ &= \delta_{\rm F}\delta_{\rm F}^\dagger\left(\partial_\mu\varphi_1^\dagger\partial^\mu\varphi_1 - m_1^2\varphi_1^\dagger\varphi_1 + \partial_\mu\varphi_2^\dagger\partial^\mu\varphi_2 - m_2^2\varphi_2^\dagger\varphi_2 - \lambda\varphi_1^\dagger\varphi_1\varphi_2^\dagger\varphi_2\right), \end{aligned} \tag{44}$$

where we take $q=1$ for simplicity. We find that $\mathscr{L}_{\varphi_i,c_{\varphi_i}}$ does not receive any radiative corrections, due to the cancellation between contributions from φ_i and c_{φ_i}, in the presence of interactions. Or $Q_{\rm F}$-doublets interact with each other respecting the $OSp(2|2)$ invariance at the quantum level. This system is also unrealistic, because all fields become unphysical and only the vacuum state survives as a physical one after imposing subsidiary conditions on states.

3 Systems of spinor fields with fermionic symmetries

We study the system that an ordinary spinor field ψ and its bosonic counterpart c_ψ coexist, described by the Lagrangian density,

$$\mathscr{L}_{\psi,c_\psi} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi + i\overline{c}_\psi\gamma^\mu\partial_\mu c_\psi - m\overline{c}_\psi c_\psi, \tag{45}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$, $\overline{c}_\psi \equiv c_\psi^\dagger\gamma^0$ and γ^μ are the gamma matrices satisfying $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$.

The canonical conjugate momentum of ψ and c_ψ are given by

$$\pi_\psi \equiv \left(\frac{\partial\mathscr{L}_{\psi,c_\psi}}{\partial\dot{\psi}}\right)_{\rm R} = i\overline{\psi}\gamma^0 = i\psi^\dagger,\ \ \pi_{c_\psi} \equiv \left(\frac{\partial\mathscr{L}_{c_\psi}}{\partial\dot{c}_\psi}\right)_{\rm R} = i\overline{c}_\psi\gamma^0 = ic_\psi^\dagger. \tag{46}$$

By solving the Dirac equations $(i\gamma^\mu\partial_\mu - m)\psi = 0$ and $(i\gamma^\mu\partial_\mu - m)c_\psi = 0$, we obtain the solutions,

$$\psi(x) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\sum_s\left(a(\boldsymbol{k},s)u(\boldsymbol{k},s)e^{-ikx} + b^\dagger(\boldsymbol{k},s)v(\boldsymbol{k},s)e^{ikx}\right), \tag{47}$$

$$\pi_\psi(x) = i\int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\sum_s\left(a^\dagger(\boldsymbol{k},s)u^\dagger(\boldsymbol{k},s)e^{ikx} + b(\boldsymbol{k},s)v^\dagger(\boldsymbol{k},s)e^{-ikx}\right), \tag{48}$$

$$c_\psi(x) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\sum_s\left(c(\boldsymbol{k},s)u(\boldsymbol{k},s)e^{-ikx} + d^\dagger(\boldsymbol{k},s)v(\boldsymbol{k},s)e^{ikx}\right), \tag{49}$$

$$\pi_{c_\psi}(x) = i\int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\sum_s\left(c^\dagger(\boldsymbol{k},s)u^\dagger(\boldsymbol{k},s)e^{ikx} + d(\boldsymbol{k},s)v^\dagger(\boldsymbol{k},s)e^{-ikx}\right), \tag{50}$$

where s represents the spin state, and $u(\boldsymbol{k},s)$ and $v(\boldsymbol{k},s)$ are Dirac spinors on the momentum space. They satisfy the relations,

$$\sum_s u(\boldsymbol{k},s)\overline{u}(\boldsymbol{k},s) = \not{k} + m,\ \ \sum_s v(\boldsymbol{k},s)\overline{v}(\boldsymbol{k},s) = \not{k} - m, \tag{51}$$

where $\overline{u}(\boldsymbol{k},s) \equiv u^{\dagger}(\boldsymbol{k},s)\gamma^0$, $\overline{v}(\boldsymbol{k},s) \equiv v^{\dagger}(\boldsymbol{k},s)\gamma^0$ and $\not{k} = \gamma^{\mu}k_{\mu}$.

Using (46), the Hamiltonian density is obtained as

$$\mathscr{H}_{\psi,c_\psi} = \pi_\psi\dot{\psi} + \pi_{c_\psi}\dot{c}_\psi - \mathscr{L}_{\psi,c_\psi} = -i\sum_{i=1}^{3}\overline{\psi}\gamma^i\partial_i\psi + m\overline{\psi}\psi - i\sum_{i=1}^{3}\overline{c}_\psi\gamma^i\partial_i c_\psi + m\overline{c}_\psi c_\psi. \quad (52)$$

The system is quantized by regarding variables as operators and imposing the following relations on the canonical pairs (ψ, π_ψ) and (c_ψ, π_{c_ψ}),

$$\{\psi^\alpha(\boldsymbol{x},t), \pi_\psi^\beta(\boldsymbol{y},t)\} = i\delta^{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}),\ \ [c_\psi^\alpha(\boldsymbol{x},t), \pi_{c_\psi}^\beta(\boldsymbol{y},t)] = i\delta^{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}), \quad (53)$$

and others are zero. Here, α and β are spinor indices. Or equivalently, the following relations are imposed on,

$$\{a(\boldsymbol{k},s), a^\dagger(\boldsymbol{l},s')\} = \delta_{ss'}\delta^3(\boldsymbol{k}-\boldsymbol{l}),\ \ \{b(\boldsymbol{k},s), b^\dagger(\boldsymbol{l},s')\} = \delta_{ss'}\delta^3(\boldsymbol{k}-\boldsymbol{l}), \quad (54)$$
$$[c(\boldsymbol{k},s), c^\dagger(\boldsymbol{l},s')] = \delta_{ss'}\delta^3(\boldsymbol{k}-\boldsymbol{l}),\ \ [d(\boldsymbol{k},s), d^\dagger(\boldsymbol{l},s')] = -\delta_{ss'}\delta^3(\boldsymbol{k}-\boldsymbol{l}), \quad (55)$$

and others are zero.

By inserting (47) – (50) into (52), the Hamiltonian H_{ψ,c_ψ} is written by

$$\begin{aligned} H_{\psi,c_\psi} = \int \mathscr{H}_{\psi,c_\psi} d^3x = \int d^3k \sum_s k_0 \Big(& a^\dagger(\boldsymbol{k},s)a(\boldsymbol{k},s) + b^\dagger(\boldsymbol{k},s)b(\boldsymbol{k},s) \\ & + c^\dagger(\boldsymbol{k},s)c(\boldsymbol{k},s) - d^\dagger(\boldsymbol{k},s)d(\boldsymbol{k},s)\Big), \end{aligned} \quad (56)$$

where the sum of the zero point energies vanishes due to the cancellation between contributions from $(\psi, \psi^\dagger)$ and $(c_\psi, c_\psi^\dagger)$.

The eigenstates for H_{ψ,c_ψ} are constructed by acting the creation operators $a^\dagger(\boldsymbol{k},s)$, $b^\dagger(\boldsymbol{k},s)$, $c^\dagger(\boldsymbol{k},s)$ and $d^\dagger(\boldsymbol{k},s)$ on the vacuum state $|0\rangle$, where $|0\rangle$ is defined by the conditions $a(\boldsymbol{k},s)|0\rangle = 0$, $b(\boldsymbol{k},s)|0\rangle = 0$, $c(\boldsymbol{k},s)|0\rangle = 0$ and $d(\boldsymbol{k},s)|0\rangle = 0$. The energy is positive semi-definite, because the effect on the negative sign in front of $d^\dagger(\boldsymbol{k},s)d(\boldsymbol{k},s)$ in H_{φ,c_φ} changes into an opposite one by the negative sign in the relation $[d(\boldsymbol{k},s), d^\dagger(\boldsymbol{l},s)] = -\delta_{ss'}\delta^3(\boldsymbol{k}-\boldsymbol{l})$.

We find that two fields separated by a space-like interval anti-commute or commute with each other as seen from $\varDelta(x-y) = 0$ for $(x-y)^2 < 0$ and the relations,

$$\begin{aligned} \{\psi^\alpha(x), \overline{\psi}^\beta(y)\} = [c_\psi^\alpha(x), \overline{c}_\psi^\beta(y)] &= \left(i\gamma^\mu\partial_\mu + m\right)^{\alpha\beta}\int \frac{d^3k}{(2\pi)^3 2k_0}\left(e^{-ik(x-y)} - e^{ik(x-y)}\right) \\ &= \left(i\gamma^\mu\partial_\mu + m\right)^{\alpha\beta} i\varDelta(x-y) \equiv iS^{\alpha\beta}(x-y), \end{aligned} \quad (57)$$

$$\{\psi^\alpha(x), \psi^\beta(y)\} = 0,\ \ \{\overline{\psi}^\alpha(x), \overline{\psi}^\beta(y)\} = 0,\ \ [c_\psi^\alpha(x), c_\psi^\beta(y)] = 0,\ \ [\overline{c}_\psi^\alpha(x), \overline{c}_\psi^\beta(y)] = 0, \quad (58)$$

$$[\psi^\alpha(x), c_\psi^\beta(y)] = 0,\ \ [\psi^\alpha(x), \overline{c}_\psi^\beta(y)] = 0,\ \ [\overline{\psi}^\alpha(x), c_\psi^\beta(y)] = 0,\ \ [\overline{\psi}^\alpha(x), \overline{c}_\psi^\beta(y)] = 0. \quad (59)$$

Hence, the microscopic causality also holds on.

The system contains negative norm states as seen from the relation $[d(\boldsymbol{k},s), d^\dagger(\boldsymbol{l},s')] = -\delta_{ss'}\delta^3(\boldsymbol{k}-\boldsymbol{l})$. It is also shown that the system has fermionic symmetries and they can guarantee the unitarity of the system.

The $\mathscr{L}_{\psi,c_\psi}$ is invariant under the fermionic transformations,

$$\delta_{\mathrm{F}}\psi = r\zeta c_\psi,\ \ \delta_{\mathrm{F}}\psi^\dagger = 0,\ \ \delta_{\mathrm{F}}c_\psi = 0,\ \ \delta_{\mathrm{F}}c_\psi^\dagger = r\zeta\psi^\dagger, \tag{60}$$

$$\delta_{\mathrm{F}}^\dagger\psi = 0,\ \ \delta_{\mathrm{F}}^\dagger\psi^\dagger = r\zeta^\dagger c_\psi^\dagger,\ \ \delta_{\mathrm{F}}^\dagger c_\psi = -r\zeta^\dagger\psi,\ \ \delta_{\mathrm{F}}^\dagger c_\psi^\dagger = 0 \tag{61}$$

and the $U(1)$ transformation,

$$\delta\psi = -iq\epsilon\psi,\ \ \delta\psi^\dagger = iq\epsilon\psi^\dagger,\ \ \delta c_\psi = -iq\epsilon c_\psi,\ \ \delta c_\psi^\dagger = iq\epsilon c_\psi^\dagger, \tag{62}$$

where $r = q^{1/2}$ and q is the $U(1)$ charge of ψ and c_ψ. The corresponding generators are given by

$$Q_{\mathrm{F}} = -i\int d^3k \sum_s r\left(a^\dagger(\boldsymbol{k},s)c(\boldsymbol{k},s) - d^\dagger(\boldsymbol{k},s)b(\boldsymbol{k},s)\right), \tag{63}$$

$$Q_{\mathrm{F}}^\dagger = -i\int d^3k \sum_s r\left(c^\dagger(\boldsymbol{k},s)a(\boldsymbol{k},s) - b^\dagger(\boldsymbol{k},s)d(\boldsymbol{k},s)\right), \tag{64}$$

$$N_{\mathrm{D}} = \int d^3k \sum_s q\Big(a^\dagger(\boldsymbol{k},s)a(\boldsymbol{k},s) - b^\dagger(\boldsymbol{k},s)b(\boldsymbol{k},s) + c^\dagger(\boldsymbol{k},s)c(\boldsymbol{k},s) + d^\dagger(\boldsymbol{k},s)d(\boldsymbol{k},s)\Big). \tag{65}$$

We have the algebraic relations ${Q_{\mathrm{F}}}^2 = 0$, ${Q_{\mathrm{F}}^\dagger}^2 = 0$ and $\{Q_{\mathrm{F}}, Q_{\mathrm{F}}^\dagger\} = N_{\mathrm{D}}$. We find that the $U(1)$ charge of particle corresponding $b^\dagger(\boldsymbol{k},s)|0\rangle$ is opposite to that corresponding $a^\dagger(\boldsymbol{k},s)|0\rangle$. Hence, $a(\boldsymbol{k},s)$ and $b^\dagger(\boldsymbol{k},s)$ are regarded as the annihilation operator of particle and the creation operator of antiparticle, respectively. In the same way, $c(\boldsymbol{k},s)$ and $d^\dagger(\boldsymbol{k},s)$ are regarded as the annihilation operator of bosonic particle and the creation operator of bosonic antiparticle, respectively.

It is easily understood that $\mathscr{L}_{\psi,c_\psi}$ is invariant under the transformations (60) and (61), from the nilpotency of $\boldsymbol{\delta}_{\mathrm{F}}$ and $\boldsymbol{\delta}_{\mathrm{F}}^\dagger$ and the relations,

$$\mathscr{L}_{\psi,c_\psi} = \boldsymbol{\delta}_{\mathrm{F}}\boldsymbol{\delta}_{\mathrm{F}}^\dagger\left(\mathscr{L}_\psi/q\right) = -\boldsymbol{\delta}_{\mathrm{F}}^\dagger\boldsymbol{\delta}_{\mathrm{F}}\left(\mathscr{L}_\psi/q\right), \tag{66}$$

where $\mathscr{L}_\psi$ is given by

$$\mathscr{L}_\psi = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi. \tag{67}$$

The Hamiltonian density $\mathscr{H}_{\psi,c_\psi}$ is written in the Q_{F} and $Q_{\mathrm{F}}^\dagger$ exact forms such that

$$\mathscr{H}_{\psi,c_\psi} = \left\{Q_{\mathrm{F}}, \left\{Q_{\mathrm{F}}^\dagger, \mathscr{H}_\psi/q\right\}\right\} = -\left\{Q_{\mathrm{F}}^\dagger, \left\{Q_{\mathrm{F}}, \mathscr{H}_\psi/q\right\}\right\}, \tag{68}$$

where $\mathscr{H}_\psi$ is given by

$$\mathscr{H}_\psi = -i\sum_{i=1}^{3}\overline{\psi}\gamma^i\partial_i\psi + m\overline{\psi}\psi. \tag{69}$$

To formulate our model in a consistent manner, we impose the subsidiary conditions,

$$Q_{\mathrm{F}}|\mathrm{phys}\rangle = 0,\ \ Q_{\mathrm{F}}^\dagger|\mathrm{phys}\rangle = 0,\ \ N_{\mathrm{D}}|\mathrm{phys}\rangle = 0, \tag{70}$$

and find that all states, except for the vacuum state $|0\rangle$, are unphysical through the quartet mechanism, in the similar way as the scalar fields in the previous section.

4 Conclusions and discussions

We have studied the quantization of systems that contain both ordinary fields with a positive norm and their counterparts obeying different statistics, and found that the systems have new type of fermionic symmetries and the unitarity of systems holds by imposing subsidiary conditions on states.

The systems considered are unrealistic, because they are empty leaving the vacuum state alone as the physical state. Q_{F} singlet fields are needed to realize our world. For a system that Q_{F}-singlets and Q_{F}-doublets coexist with exact fermionic symmetries, the Lagrangian density is, in general, written in the form as $\mathscr{L}_{\mathrm{Total}} = \mathscr{L}_{\mathrm{S}} + \mathscr{L}_{\mathrm{D}} + \mathscr{L}_{\mathrm{mix}} = \mathscr{L}_{\mathrm{S}} + \boldsymbol{\delta}_{\mathrm{F}}\boldsymbol{\delta}_{\mathrm{F}}^{\dagger}(\Delta\mathscr{L})$. Here, $\mathscr{L}_{\mathrm{S}}$, $\mathscr{L}_{\mathrm{D}}$ and $\mathscr{L}_{\mathrm{mix}}$ stand for the Lagrangian density for Q_{F}-singlets, Q_{F}-doublets and interactions between Q_{F}-singlets and Q_{F}-doublets. Under the subsidiary conditions $Q_{\mathrm{F}}|\mathrm{phys}\rangle = 0$, $Q_{\mathrm{F}}^{\dagger}|\mathrm{phys}\rangle = 0$ and $N_{\mathrm{D}}|\mathrm{phys}\rangle = 0$ on states, all Q_{F}-doublets become unphysical. This system seems to be same as that described by $\mathscr{L}_{\mathrm{S}}$ alone, because Q_{F}-doublets do not give any dynamical effects on Q_{F}-singlets. From this, we suppose that it is not possible to show the existence of Q_{F}-doublets. However, in a very special case, an indirect proof would be possible through fingerprints left by symmetries in a fundamental theory. The fingerprints are specific relations among parameters such as a unification of coupling constants, reflecting on underlying symmetries [17]. This subject will be reexamined in the separate publication [26].

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A $OSp(2|2)$ and $OSp(1,1|2)$

The $OSp(2|2)$ is the group whose elements are generators of transformations which leave the inner product $x^2 + y^2 + 2i\theta_1\theta_2$. Here, x and y are real numbers, and θ_1 and θ_2 are hermitian Grassmann numbers,

$$\theta_1^{\dagger} = \theta_1,\ \ \theta_2^{\dagger} = \theta_{,}\ \ {\theta_1}^2 = 0,\ \ {\theta_2}^2 = 0. \tag{71}$$

The infinitesimal transformations are classified into following types.
(a) Rotation relating x and y:

$$\delta_{\mathrm{r}} x = -\epsilon_{\mathrm{r}} y,\ \ \delta_{\mathrm{r}} y = \epsilon_{\mathrm{r}} x,\ \ \delta_{\mathrm{r}}\theta_1 = 0,\ \ \delta_{\mathrm{r}}\theta_2 = 0, \tag{72}$$

where ϵ_{r} is an infinitesimal real parameter.
(b) Rotation relating θ_1 and θ_2:

$$\delta_{\mathrm{r}'} x = 0,\ \ \delta_{\mathrm{r}'} y = 0,\ \ \delta_{\mathrm{r}'}\theta_1 = -\epsilon_{\mathrm{r}'}\theta_2,\ \ \delta_{\mathrm{r}'}\theta_2 = \epsilon_{\mathrm{r}'}\theta_1, \tag{73}$$

where $\epsilon_{\mathrm{r}'}$ is an infinitesimal real parameter.
(c) Fermionic transformations:

$$\delta_1 x = -i\zeta_1\theta_2,\ \ \delta_1 y = i\zeta_1\theta_1,\ \ \delta_1\theta_1 = \zeta_1 x,\ \ \delta_1\theta_2 = \zeta_1 y, \tag{74}$$

$$\delta_2 x = -i\zeta_2\theta_1,\ \ \delta_2 y = -i\zeta_2\theta_2,\ \ \delta_2\theta_1 = \zeta_2 y,\ \ \delta_2\theta_2 = -\zeta_2 x, \tag{75}$$

where ζ_1 and ζ_2 are Grassmann numbers.

By introducing four hermitian scalar fields, we can construct a Lagrangian density with $OSp(2|2)$ invariance as follows,

$$\mathscr{L}_{OSp(2|2)} = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) - \frac{1}{2}m^2\left({\phi_1}^2 + {\phi_2}^2\right) + i\partial_\mu c_1\partial^\mu c_2 - im^2 c_1 c_2, \tag{76}$$

where ϕ_1 and ϕ_2 are ordinary hermitian scalar fields and c_1 and c_2 are fermionic hermitian scalar fields.

Using complex scalar fields defined by

$$\varphi \equiv \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right),\ \ c_\varphi \equiv \frac{1}{\sqrt{2}}\left(c_1 + ic_2\right), \tag{77}$$

the above Lagrangian density (76) is rewritten as

$$\mathscr{L}_{OSp(2|2)} = \partial_\mu\varphi^\dagger\partial^\mu\varphi - m^2\varphi^\dagger\varphi + \partial_\mu c_\varphi^\dagger\partial^\mu c_\varphi - m^2 c_\varphi^\dagger c_\varphi. \tag{78}$$

The Lagrangian density (78) is just given by (1).

For a reference sake, we compare the above-mensioned system with a system of scalar fields with $OSp(1,1|2)$. The $OSp(1,1|2)$ is the group whose elements are generators of transformations which leave the inner product $x^2 - y^2 + 2i\theta_1\theta_2$. Notice that a negative sign exists in front of y^2. The infinitesimal transformations are classified into following types.
(a) Boost relating x and y:

$$\delta_{\mathrm{b}} x = -\epsilon_{\mathrm{b}} y,\ \ \delta_{\mathrm{b}} y = -\epsilon_{\mathrm{b}} x,\ \ \delta_{\mathrm{b}}\theta_1 = 0,\ \ \delta_{\mathrm{b}}\theta_2 = 0, \tag{79}$$

where ϵ_{b} is an infinitesimal real parameter.
(b) Rotation relating θ_1 and θ_2:

$$\delta_{\mathrm{r}'} x = 0,\ \ \delta_{\mathrm{r}'} y = 0,\ \ \delta_{\mathrm{r}'}\theta_1 = -\epsilon_{\mathrm{r}'}\theta_2,\ \ \delta_{\mathrm{r}'}\theta_2 = \epsilon_{\mathrm{r}'}\theta_1, \tag{80}$$

where $\epsilon_{\mathrm{r}'}$ is an infinitesimal real parameter.
(c) Fermionic transformations:

$$\delta_{\mathrm{B}} x = \lambda\theta_1,\ \ \delta_{\mathrm{B}} y = -\lambda\theta_1,\ \ \delta_{\mathrm{B}}\theta_1 = 0,\ \ \delta_{\mathrm{B}}\theta_2 = i\lambda(x+y), \tag{81}$$

$$\overline{\delta}_{\mathrm{B}} x = \lambda\theta_2,\ \ \overline{\delta}_{\mathrm{B}} y = -\lambda\theta_2,\ \ \overline{\delta}_{\mathrm{B}}\theta_1 = -i\lambda(x+y),\ \ \overline{\delta}_{\mathrm{B}}\theta_2 = 0, \tag{82}$$

where λ is a Grassmann numbers with $\lambda^* = -\lambda$.

By introducing four hermitian scalar fields, we can construct a Lagrangian density with $OSp(1,1|2)$ invariance as follows,

$$\mathscr{L}_{OSp(1,1|2)} = \frac{1}{2}\left(\partial_\mu\phi_3\partial^\mu\phi_3 - \partial_\mu\phi_0\partial^\mu\phi_0\right) - \frac{1}{2}m^2\left({\phi_3}^2 - {\phi_0}^2\right) + i\partial_\mu c_1\partial^\mu c_2 - im^2 c_1 c_2, \tag{83}$$

where ϕ_3 is an ordinary hermitian scalar field, ϕ_0 is a hermitian scalar field with a negative norm, and c_1 and c_2 are fermionic hermitian scalar fields.

Using hermitian scalar fields defined by

$$B \equiv \frac{1}{\sqrt{2}}\left(\phi_3+\phi_0\right), \ \ \phi \equiv \frac{1}{\sqrt{2}}\left(\phi_3-\phi_0\right), \tag{84}$$

the above Lagrangian density (83) is rewritten as

$$\mathscr{L}_{OSp(1,1|2)} = \partial_\mu B \partial^\mu \phi - m^2 B\phi + i\partial_\mu \overline{c}\partial^\mu c - im^2\overline{c}c, \tag{85}$$

where $\overline{c} = c_1$ and $c = c_2$. The interacting model containing $\mathscr{L}_{OSp(1,1|2)}$ as a free part has been constructed and studied [14, 15].

The Lagrangian density (85) is invariant under the following fermionic transformations,

$$\delta_{\mathrm{B}}\phi = \lambda c, \ \ \delta_{\mathrm{B}}c = 0, \ \ \delta_{\mathrm{B}}\overline{c} = i\lambda B, \ \ \delta_{\mathrm{B}}B = 0, \tag{86}$$

$$\overline{\delta}_{\mathrm{B}}\phi = \lambda\overline{c}, \ \ \overline{\delta}_{\mathrm{B}}c = -i\lambda B \ \ \overline{\delta}_{\mathrm{B}}\overline{c} = 0, \ \ \overline{\delta}_{\mathrm{B}}B = 0. \tag{87}$$

They correspond to the BRST and anti-BRST transformations, respectively. The following algebraic relations hold:

$${Q_{\mathrm{B}}}^2 = 0, \ \ {\overline{Q}_{\mathrm{B}}}^2 = 0, \ \ \{Q_{\mathrm{B}}, \overline{Q}_{\mathrm{B}}\} = 0, \tag{88}$$

where Q_{B} and $\overline{Q}_{\mathrm{B}}$ are the BRST and the anti-BRST charges given by

$$\delta_{\mathrm{B}}\Phi = i[\lambda Q_{\mathrm{B}}, \Phi], \ \ \overline{\delta}_{\mathrm{B}}\Phi = i[\lambda\overline{Q}_{\mathrm{B}}, \Phi]. \tag{89}$$

The Lagrangian density (85) is rewritten by

$$\mathscr{L}_{OSp(1,1|2)} = \boldsymbol{\delta}_{\mathrm{B}}\left(-i\partial_\mu\overline{c}\partial^\mu\phi + im^2\overline{c}\phi\right) = \boldsymbol{\delta}_{\mathrm{B}}\overline{\boldsymbol{\delta}}_{\mathrm{B}}\left(-\frac{i}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}m^2\phi\phi\right), \tag{90}$$

where $\boldsymbol{\delta}_{\mathrm{B}}$ and $\overline{\boldsymbol{\delta}}_{\mathrm{B}}$ are defined by $\delta_{\mathrm{B}} = \lambda\boldsymbol{\delta}_{\mathrm{B}}$ and $\overline{\delta}_{\mathrm{B}} = \lambda\overline{\boldsymbol{\delta}}_{\mathrm{B}}$, respectively.

Finally, we point out that the Lagrangian density (90) consists of the gauge-fixing term and the Faddeev-Popov ghost term for the system of ϕ with an empty dynamics. The system with the empty action integral $S = 0$ has the invariance under the local transformation $\phi(x) \to \phi_\Lambda = \phi(x) + \Lambda(x)$, and after taking the gauge-fixing condition,

$$f(\phi_\Lambda(x)) = (\partial_\mu\partial^\mu + m^2)\phi(x) = 0, \tag{91}$$

we obtain the Lgrangian density,

$$\mathscr{L}_{\mathrm{gf+gh}} = \boldsymbol{\delta}_{\mathrm{B}}\left(-i\overline{c}(\partial_\mu\partial^\mu + m^2)\phi\right) = -B(\partial_\mu\partial^\mu + m^2)\phi - i\overline{c}(\partial_\mu\partial^\mu + m^2)c. \tag{92}$$

The Lagrangian density (92) becomes $\mathscr{L}_{OSp(1,1|2)}$ after the partial integration in the action integral.

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